PATTERNS IN THE PLANE AND BEYOND: SYMMETRY IN TWO AND THREE DIMENSIONS

B. G. THOMAS AND M. A. HANN
PATTERNS IN THE PLANE AND BEYOND:
SYMMETRY IN TWO AND THREE DIMENSIONS

by
B.G. Thomas and M.A. Hann

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patterns in the plane and beyond: symmetry in two and three dimensions.

Authors: B.G. Thomas and M.A. Hann
Foreword: D. Holdcroft and C. Hammond

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Foreword
Professor D. Holdcroft, Chairman of ULITA Committee, and
Dr C. Hammond, Leeds Philosophical and Literary Society.

Foreword by D. Holdcroft
This monograph has been produced as an accompaniment to the
exhibition ‘Form, Shape and Space: An Exhibition of Tilings and
Polyhedra’. The exhibition is composed largely of items designed and
produced by B.G. Thomas as part of a doctoral project supervised by
M.A. Hann.

This monograph examines a range of geometric concepts of importance
to the further understanding of two- and three-dimensional designs.
A brief explanation is given of symmetry in patterns and attention
is focused on a particular set of polyhedra, known as the regular or
Platonic solids. The principal focus is on developing the means by
which these solid forms can be decorated with regular patterns, in
such a way that gap and overlap is avoided and precise registration is
a feature.

The patterning of regular solids in ways which ensure precise
registration and the absence of gaps or overlaps is not as straight
forward a matter as it may initially seem. When regular repeating
patterns, which perform to satisfaction on the Euclidean plane, are
folded several times, into different planes, their component parts will
not correspond readily. Only certain pattern types, with particular
symmetry characteristics, are suited to the precise patterning of
each Platonic solid. This monograph presents a systematic means
by which appropriate pattern types can be identified. The symmetry
characteristics of importance to the process are identified, and the
patterning of each of the five solids is explained and illustrated, and
a set of rules is proposed. The monograph is illustrated fully using
original illustrative material derived from the images, patterns and
designs which feature in the exhibition.
Foreword by C. Hammond

For much of the twentieth century, the University of Leeds played a pivotal role in the analysis and interpretation of patterns – the three-dimensional patterns which are the basis of crystal structures and the two-dimensional patterns which are the basis of fabric designs, tessellations and tilings. This role may be said to have begun with the Nobel Prize-winning work of W.H. Bragg, Cavendish Professor of Physics, and his son W.L. Bragg. Working as a team, using x-ray diffraction techniques, they solved the first crystal structures in 1913. In the 1930s H.J. Woods of the Department of Textile Industries presented a comprehensive appraisal of symmetries in patterns. Drawing on concepts which have their origin in the study of crystal structures, Woods was the first to present the complete and explicit enumeration of the two-colour one- and two-dimensional patterns, visionary work which was several years ahead conceptually of the theoretical developments emanating from crystallographers worldwide. Today it is acknowledged widely that Woods helped to lay the foundation for our current thinking on the geometry of regular repeating patterns and tilings and, in particular, our knowledge of colour (counterchange) symmetry. Between the 1930s and 1940s W.T. Astbury, building on work initiated by J.B. Speakman, also of the Department of Textile Industries at Leeds, pioneered the use of x-ray diffraction techniques in the quest to understand the structure of wool fibre, work which (it could be argued) led directly to the discovery of the structure of DNA. Indeed Astbury, together with his research student Florence Bell, in 1938, took the first x-ray diffraction photographs of DNA.

The Leeds tradition continues and contributions have been made to furthering the understanding of pattern geometry within different cultural and historical contexts; identifying concepts of importance to pattern classification; elucidating the principles underlying counterchange patterns; exploring concepts in the field of layer symmetry, thus advancing our understanding of the geometry of woven textiles.
One particular project has the aim of developing a systematic means by which all-over patterns can be used to cover (or pattern) regular polyhedra. This monograph and the exhibition it accompanies are the initial outcome of this research; both confirm that Leeds continues at the forefront of research into structure and form.
1. Introduction

Symmetry pervades our everyday lives and environment. We live in a symmetrical world. We wear clothes which are symmetrical. We live and work in buildings which are largely symmetrical. We drive automobiles which are symmetrical. In fact the vast majority of living creatures, manufactured objects, constructions, monuments, tools, implements and utensils exhibit bi-lateral symmetry. This is where two component and equal parts are each a reflection of the other. The meaning of the term symmetry can be extended beyond this everyday use to include other geometrical actions and their combinations; in all cases the essence is one of regular reproduction or repetition of a fundamental unit, shape, figure or other element. These further geometric actions are known as “symmetry operations” or “symmetries”, and are most readily understood if considered, at least initially, in a two-dimensional context, although it should be noted that many of the relevant concepts were developed in association with the improved understanding of three-dimensional crystal structures in the late-nineteenth and twentieth centuries. This monograph reviews the historical developments in the study of patterns and, in particular, how our understanding has been enhanced through adopting concepts and principles sourced in mathematics and the discipline of crystallography. The symmetry rules governing both two-dimensional patterns and three-dimensional polyhedra are outlined. Finally consideration is given to how patterns which exist in the plane can be applied to repeat regularly around the surface of polyhedra.
2. The Study of Pattern Geometry: Historical Precedents

The academic study of pattern is of relatively recent origin. Arguably the most influential nineteenth-century European study is Owen Jones’ *The Grammar of Ornament* [1856], which focused on the cultural and historical principles of patterned ornament. This work is of importance as it is the first comprehensive attempt to categorise world patterns with reference to culture and period of origin. Geometric considerations came a few decades later, when certain authors focused on pattern construction. Indeed, some observers exhibited an astute awareness of the fundamental principles underlying the construction of all-over patterns. Meyer, for example, grouped designs according to spatial characteristics into enclosed spaces, ribbon-like bands, or unlimited flat patterns, corresponding to motifs, border patterns and all-over patterns respectively [Meyer, 1894, p.3]. He recognised also that in the context of these unlimited flat patterns there was a ‘...certain division, a subsidiary construction or a network’, thus anticipating the realization that lattice structures are an underlying structural feature of all-over patterns. A few years later, Stevenson and Suddards [1897, chps. 2-5], in their appraisal of the geometry of Jacquard-woven patterns, illustrated constructions based on rectangular, rhombic, hexagonal and square lattices. Similarly, Day [1903] emphasised the geometrical construction of patterns, illustrating all-over patterns based on square, parallelogram, rhombic and hexagonal type lattices. Christie’s *Pattern Design*, first published in 1910, made a formal ordering of patterns according to the nature of their constituent motifs, rather than grouping them according to time periods or cultural context. Christie’s work is of significance as it represents an early stage in the categorisation of patterns in terms of their geometric properties. The
early-twentieth century also saw the evolution of another perspective of pattern analysis and classification: the consideration of patterns by reference to their symmetry characteristics. Relevant concepts have their origin in the scientific investigation of crystals, pioneered largely by the work of Fedorov, the Russian crystallographer who, in the late-nineteenth century, determined that there were 230 three-dimensional crystallographic groups before proving that regularly repeating patterns of the plane are constructed in accordance with the seventeen crystallographic symmetry groups [Fedorov, 1885 and 1891, p.345, cited by Grünbaum and Shephard, 1987, p.55]. This theorem was rediscovered in 1897 by Fricke and Klein [1897, pp.227-233; Martin, 1982, p.111], but since the focus of crystallographers was primarily towards higher-dimensional phenomena it was not until the 1920s that interest in the enumeration of the two-dimensional crystallographic groups was aroused through the work of Pólya and Niggli [1924].

Developments in the application of crystallographic theory continued and in 1933 Birkhoff defined and illustrated the four symmetry operations, recognizing their occurrence in motifs and patterns [Birkhoff, 1933]. In the context of design, and in particular textiles, an important early attempt to classify regularly repeating patterns according to their underlying geometry was made by physicist H.J. Woods in the 1930s [Woods, 1935a, b, c and 1936]. Drawing on concepts, which have their origin in the study of the crystal structures of molecules, Woods presented a comprehensive appraisal of symmetry in patterns. In fact, Woods was the first to present the complete and explicit enumeration of the two-colour, one- and two-dimensional patterns (i.e. two colour counter-change border and all-over patterns). This visionary work was several years ahead conceptually of the theoretical developments emanating from crystallographers worldwide. Unfortunately this work went largely unnoticed until nearly forty years later when Branko Grünbaum recognised its relevance to contemporary research on coloured tilings and brought it to the attention of mathematicians [Crowe, 1986, p.408]. Today, it is acknowledged widely that Woods helped to lay the foundation for our current thinking on the geometrical characteristics of regular repeating patterns [Washburn and Crowe, 1988].
Buerger and Lukesh [1937] made a notable study which presented a series of symbols denoting lattice structures, orders of rotation, and the presence of reflection and glide-reflection axes. (These and other related geometrical features are explained in later sections of this monograph.) Brainerd initiated the use of symmetry as a tool for the classification of archaeological artefacts in 1942 and, in 1952, Weyl presented a review of symmetry in art, botany and other pure sciences. Russian crystallographers, Shubnikov and Koptsik, also provided new perspectives on symmetry [Shubnikov and Koptsik, 1974]. The works of Walker and Padwick [1977], Schattschneider [1978, 1986] and Stevens [1984] are probably the most readily accessible to the non-scientist. In 1980, Crowe presented a recognition chart to aid identification of the seventeen classes of all-over patterns. This was developed further in collaboration with Washburn to account also for two-colour-counterchange possibilities [Crowe and Washburn, 1986]. Grünbaum and Shephard charted much of what is currently known on the subject of regular tilings in their extensive work published in 1987, a landmark in the mathematical investigation of patterns and tilings. It should be noted that often the words patterns and tilings are used interchangeably in the literature. Strictly speaking, patterns are those designs comprised of a motif (or motifs) which are set against a background, and tilings are those designs which cover the plane without gap or overlap and do not comprise a background component. The word pattern (or more precisely all-over pattern) is the noun of choice throughout this monograph.

Washburn and Crowe [1988] published an impressive treatise dealing with the theory and practice of pattern analysis, using symmetry in the analysis of designs from different cultures. This has proved a classic reference for anthropologists, archaeologists, art historians, mathematicians and designers. A more recent publication by the same authors, developed the perspectives presented previously and examined how cultures use patterns to encode meaning [Washburn and Crowe, 2004]. The extensive bibliography also updated its predecessor. Hargittai [ed., 1986, 1989] published two compendia containing over one hundred papers from the sciences, arts and humanities, dealing with the nature of geometric symmetry and its
occurrence and application. Schattschneider’s monumental study of the work of artist M.C. Escher, provided not only an insight into the periodic drawings of the artist, but also explained symmetry in ways which made the relevant concepts understandable to a general audience of non-mathematicians [Schattschneider, 1990]. Hargittai and Hargittai [1994] published a profusely illustrated review of the principles of symmetry aimed at enhancing understanding among non-specialists.

Kapraff [1991] recognised the possibilities offered by geometric symmetry in the fields of mathematics, science and art. He also acknowledged the reluctance that designers often feel if expected to work within the perceived rigidity of mathematical group theory, stating that:

Symmetry is a concept that has inspired the creative work of artist and scientist; it is the common root of artistic endeavour. To an artist or architect symmetry conjures up feelings of order, balance and harmony and an organic relation between the whole and its parts. On the other hand, making these notions useful to a mathematician or scientist requires definition. Although such a definition may make the idea of symmetry less flexible than the artist’s intuitive feeling for it, that precision can actually help designers unravel the complexities of design and see greater possibilities for symmetry in their own work.

Kapraff, 1991, p. 405

In recent decades, research at the University of Leeds has been concerned with aspects of patterns and their structures. Attention has been focused on historical and cultural aspects of pattern geometry [Hann, 1992], the analysis and construction of counterchange patterns [Hann and Lin, 1995], and the geometry of woven fabric [Scivier and Hann, 2000 a, b]. A particular concern has been with the development of teaching material which allows design geometry, including pattern symmetry, to form a significant component of the curriculum delivered to design students [relevant publications include Hann and Thomson, 1992; Horne and Hann, 1998; Hann, 2003 a,b,c; Hann and Thomas, 2007]. Other work has recognized the potential value of symmetry, and other geometric concepts associated with design, as
problem-solving tools in the twenty-first century [Hann and Russell, 2003; Hann and Thomas, 2005].

The presence of symmetry in nature has fascinated both scientists and artists. Biologist and philosopher Ernst Heinrich Haeckel made detailed studies of microscopic life forms exhibiting unusual symmetric characteristics. Haeckel’s 1862 Challenger Monograph on Radiolaria illustrated over 4,000 species, several of which displayed fully triangulated skeletons [Haeckel, 1904]. In 1940, French structural innovator Lericolaris proposed a geodesic shell structure based on triangulated networks of radiolaria. Fascinated by these tiny sea creatures, he observed:

You can’t just convert these things into building structures, but there is much to admire and understand. You see some kind of coherence and a purity of design which is amazing, which is frightening.

Lericolaris, 1940, cited by Powers, 1999, p.37

Buckminster Fuller’s independent innovation of the geodesic dome dates from 1948 and displays a similar structure to many radiolaria [Pearce, 1978, p.18]. The design of geodesic dome structures ensures that the triangle system uses minimum materials in construction and it seems also that such structures become stronger, lighter and cheaper per unit volume as their size increases [Hann and Russell, 2003].

Buckminster Fuller’s ideas on geodesic structures may have stimulated significant scientific developments, such as the discovery in 1985 of a superstable all-carbon C_{60} molecule, appropriately named buckminsterfullerene [Hargittai, 1992, p.xv]. Comprised of 20 regular hexagons and 12 regular pentagons (formally known as a truncated icosahedron) and similar in shape to a soccer ball, C_{60} buckminsterfullerene has unique mechanical and electrical properties attributable to its structural geometry [Baldwin, 1996, p.74; Hann and Russell, 2003]. Hexagons and pentagons of carbon atoms link together to form a hollow geodesic globe, with bonding strains equally distributed among the 60 carbon atoms. Variants of this form, collectively known
as fullerences, have been the subject of intense research as they offer great potential as new materials in various branches of engineering [Curl and Smalley, 1991; Stewart, 2001].

Four symmetry operations are of importance in the context of two-dimensional designs: translation, rotation, reflection and glide-reflection (shown schematically in Figure 1).

![Figure 1: The four symmetry operations](image)

Key:
- translation axis
- two-fold rotation
- reflection axis
- glide-reflection axis

Translation allows a motif to undergo repetition vertically, horizontally, or diagonally at regular intervals while retaining the same orientation. Rotation allows a motif to undergo repetition at regular intervals round an imaginary fixed point (known as a centre of rotation). Reflection allows a motif to undergo repetition across an imaginary line, known as a reflection axis, producing a mirror image; this is characteristic of so-called bilateral symmetry. Glide-reflection allows a motif to be repeated in one action through a combination of translation and reflection, in association with a glide-reflection axis. Where motifs or patterns possess the same symmetry characteristics they are said to be of the same class, and may be classified accordingly; a full explanation was given by Hann and Thomson [1992].
Motifs are the building blocks from which patterns are produced. They may be either symmetrical or asymmetrical. A symmetrical motif is comprised of two or more parts, of identical size, shape and content. Depending on the constituent symmetry characteristics, motifs may be classified using the notation \( cn \) (c for cyclic) or \( dn \) (d for dihedral). Motifs from family \( cn \) have \( n \)-fold rotational symmetry and motifs from family \( dn \) have \( n \) distinct reflection axes as well as \( n \)-fold rotational symmetry. Relevant illustrations are provided in Figure 2.

![Figure 2: Classes \( cn \) and \( dn \) motifs](image)

The term pattern is used frequently to refer to surface variation or texture. More precisely a pattern exhibits an underlying regular structure, showing repetition of a motif, figure or other unit. In considering the geometry of pattern, the term symmetry may be introduced. Border patterns exhibit translation of a motif at regular intervals in one direction only, as if trapped between two imaginary parallel lines. Alternative terms include band, strip, frieze or one-dimensional patterns. Combinations of the four symmetry operations yield seven possible border pattern classes (shown in Figure 3). The notation ascribed conventionally to border patterns is of the form \( pxyz \). The letter \( p \) prefaces each of the seven. The letter \( x \) is the symbol which denotes symmetry operations perpendicular to the longitudinal axis of the border; \( m \) is used where vertical reflection is present, or the number \( r \) where the operation is absent. The third symbol, \( y \), denotes symmetry operations working parallel to the sides of the border; the letter \( m \) is used if longitudinal reflection is present, the letter \( a \) if glide-
reflection is present or the number 1 if neither is present. The fourth symbol, z, denotes the presence of two-fold rotation; the number 2 is used if rotation is present and the number 1 if rotation is not present.

![Figure 3: The seven classes of border patterns](image)

All-over patterns are characterised by translation in two independent directions across the plane. Combinations of the four symmetry operations will yield seventeen possibilities (or classes). Associated with these seventeen classes is a notation, which identifies the highest order of rotation within the pattern together with the presence (or absence) of glide-reflection and/or reflection. Where rotation is present, it may be of the order two, three, four or six, and some patterns may show combinations of these. Reflection, where it is present, may be in one or more directions, and may combine with other symmetry
operations; the same is true of glide-reflection. The seventeen all-over pattern classes are illustrated in Figure 4.

Figure 4: The seventeen classes of all-over patterns
4. The Symmetry Characteristics of All-over Patterns

As stated previously, one of the concerns of this monograph is all-over patterns, which are patterns that exhibit translation of a motif (or motifs) in two independent non-parallel directions across the plane. Synonymous terms include *wallpaper groups* [Schattschneider, 1978], *plane groups* [Stevens, 1984], *periodic patterns* [Grünbaum and Shephard, 1987], *two-dimensional patterns* [Washburn and Crowe, 1988], and *ditranslational designs* [Horne, 2000]. This section provides a review of the symmetry characteristics of all-over patterns and presents a brief explanation of the most commonly used notation.

As indicated previously, translation is the underlying symmetry feature of regularly repeating patterns and, in the case of all-over patterns, is operational in two independent directions across the plane. The introduction of one or more of the remaining three symmetry operations in association with translation, permits a total of seventeen pattern classes to be generated. Proof of the existence of only seventeen all-over pattern classes was provided by Weyl [1952], Jawson [1965], Coxeter [1969], Schwarzenberger [1974] and Martin [1982].

Geometric frameworks of corresponding points that form lattice structures are a further feature underlying the structure of all-over patterns. These lattice points form *unit cells* of identical size, shape and content which, when translated in two independent non-parallel directions, produce the full repeating pattern. There are only five distinct lattice types that may be used as generators for all-over pattern classes: parallelogram, rectangular, rhombic, square and
hexagonal, with a rhombic unit cell consisting of two equilateral triangles. Crystallographers have termed these Bravais lattices, after Bravais who was first to propose that lattices could be classified into five types [Grünbaum and Shephard, 1987, p.262]. The five geometrical lattice types are illustrated schematically in Figure 5.

![Figure 5: The five Bravais lattices](image)

Schematic illustrations of the all-over pattern classes are provided in Figure 4 and schematic diagrams of the seventeen unit cells of all-over patterns, specifying the symmetry characteristics, the conventionally selected unit cell and the fundamental region of each pattern class, are detailed in Figure 6. A key for the diagrams illustrating the symmetry characteristics of all-over patterns is given below.

Key:
- centre of two-fold rotation
- centre of three-fold rotation
- centre of four-fold rotation
- centre of six-fold rotation
- axis of reflection
- axis of glide-reflection
- translation vector
- outline of unit cell
- outline of centred cell
- fundamental region
4.1 Notation

Various notations have been used by mathematicians and crystallographers in the classification of all-over patterns; these were reviewed by Schattschneider [1978]. The most widely-accepted notation is that proposed in the *International Tables of X-Ray Crystallography* [Henry and Lonsdale, 1952]. This four-symbol notation, $p_{xyz}$ or $c_{xyz}$, indicates the type of unit cell, the highest order of rotation and the symmetry axes present in two directions. Further explanation is provided below.

The first symbol, either the letter $p$ or $c$, indicates whether the lattice cell is *primitive* or *centred*. Primitive cells are a feature of fifteen of the all-over pattern classes and can generate the full pattern by translation alone. The remaining two cells are of the rhombic lattice type with the enlarged cell containing two repeating units, one held within the centred cell, and another in quarters of the enlarged cell corners (classes $c_{1m1}$ and $c_{2mm}$, shown in Figure 6).
The second symbol, \( n \), denotes the highest order of rotation present. Where rotational symmetry is present, only two-, three-, four- and six-fold rotation axes are possible in the production of all-over patterns. Five-fold rotational symmetry across the plane is not possible. This is known as the **crystallographic restriction** [explained by Stevens, 1984, Appendix, p.376-390]. If no rotational symmetry is present, \( n = 1 \).

The third symbol represents a symmetry axis normal to the \( x \)-axis (i.e. perpendicular to the left side of the unit cell): \( m \) (for mirror) indicates a reflection axis, \( g \) (for glide) indicates a glide-reflection axis and the number \( i \) indicates that no reflection or glide-reflection axes are present normal to the \( x \)-axis.

The fourth symbol indicates a symmetry axis at angle \( \alpha \) to the \( x \)-axis, which is dependant on \( n \), the highest order of rotation. Angle \( \alpha = 180 \) degrees if \( n = 1 \) or 2, \( \alpha = 45 \) degrees if \( n = 4 \) and \( \alpha = 60 \) degrees if \( n = 3 \) or 6. The symbols \( m \) and \( g \) denote the presence of reflection and glide-reflection symmetries respectively. The absence of symbols in the third and fourth position indicates that the pattern admits no reflection or glide-reflection.

A summary of the symmetry characteristics of the seventeen classes of all-over patterns is provided in Table 1, and descriptions of each pattern class are presented subsequently. In each case the conventionally selected unit cell is stated.
### Table 1  Summary of symmetry characteristics of the seventeen all-over pattern classes

<table>
<thead>
<tr>
<th>Symmetry class</th>
<th>Lattice structure</th>
<th>Area of fundamental region/unit cell</th>
<th>Symmetry operations present</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Parallelogram</td>
<td>Rectangular</td>
<td>Square</td>
</tr>
<tr>
<td>p1</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>p2</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>p1m1</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>p1g1</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>c1m1</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>p2mm</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>p2gg</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>p2mg</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>c2mm</td>
<td>✓</td>
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<td>✓</td>
</tr>
<tr>
<td>p4</td>
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<td>p4mm</td>
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<td>p4gm</td>
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<td>p3</td>
<td>✓</td>
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<td>p3m1</td>
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<tr>
<td>p6</td>
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<tr>
<td>p6mm</td>
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</tr>
</tbody>
</table>

Source: data from Tantiwong [2000, p.27]

### 4.2 Patterns without rotational properties

There are four all-over pattern classes with no rotational symmetry, and elements of the pattern can only coincide after a full rotation of 360 degrees. From the perspective of geometrical symmetry, all-over pattern class \( p1 \) is the simplest in terms of analysis, classification
and construction. Normally constructed on a parallelogram lattice, this pattern contains no reflections, glide-reflections or rotations and repeats simply by translation of an asymmetric figure (motif class $c1$) in two independent directions. The fundamental region is of an area equivalent to the unit cell, which could be based on any of the five geometric lattice structures. The structure of a class $p1$ pattern on a square lattice is illustrated in Figure 7.

![Figure 7: Example of a $p1$ all-over pattern](image)

Class $p1m1$ all-over patterns are constructed on a rectangular or square lattice structure. The repeating pattern is created through two parallel reflections and translation in the direction of the reflection axes. The fundamental region occupies half the area of the unit cell and is bounded on opposite sides by the reflection axes. The structure of pattern class $p1m1$ is shown in Figure 8.
Based on either a rectangular or square lattice, pattern class \( p1g1 \) contains two alternating and parallel glide-reflection axes falling on the corners of the unit cell, but exhibits no reflection or rotational properties. As shown in Figure 9, the generating region is equal to half the area of the unit cell.

Class \( c1m1 \) all-over patterns are usually constructed from a rhombic lattice, although square or hexagonal lattice types may also be used. As shown in Figure 6, the pattern exhibits reflection at right angles to the enlarged cell of the rhombic lattice and parallel glide-reflection axes. The enlarged cell contains two generating regions, one contained within the centred cell and another in quarters at the enlarged cell corners. The construction of a class \( c1m1 \) tiling on a rhombic lattice is shown in Figure 10.
Figure 9: Example of a \textit{p1g1} all-over pattern

Figure 10: Example of a \textit{c1m1} all-over pattern
4.3 Patterns exhibiting two-fold rotation

There are a total of five all-over pattern classes in which the highest order of rotation is 2 (180 degree rotational symmetry): \( p2, p2mm, p2mg, p2gg \) and \( c2mm \). Each of these pattern classes appears the same when viewed upside-down or right side up. Their constituent symmetry characteristics are considered below.

Pattern class \( p2 \) may be based on any of the five geometrical lattice types, although the parallelogram lattice is the most frequently used. A total of nine points of two-fold rotation are evident: at the centre of the unit, at each of the unit corners and the mid-points of the unit sides. The fundamental region occupies half the area of the unit cell, as shown in Figure 11 which illustrates the construction of class \( p2 \) on a parallelogram lattice.

Figure 11: Example of a \( p2 \) all-over pattern
All-over pattern class \( p2mm \) is based upon either a rectangular or a square lattice, exhibiting two alternating axes of horizontal reflection and two alternating axes of vertical reflection. Centres of two-fold rotation are present where the reflection axes intersect. The generating region contained in one-quarter of the unit cell may be constructed by joining four rotational centres of the same orientation. It can be seen from Figure 12 that pattern class \( p2mm \) may be constructed through repetition (or translation) of a class \( d2 \) motif.

![Figure 12: Example of a p2mm all-over pattern](image)

Pattern class \( p2gg \) contains vertical glide-reflection axes that intersect horizontal glide-reflection axes at right angles within the unit cell, the latter being based on either a rectangular or a square lattice. The fundamental region is one-quarter of the unit cell and two-fold rotational points are positioned at the centre of the unit, at each of the unit corners and the midpoints of the unit sides. Figure 13 illustrates the construction of pattern class \( p2gg \) on a square lattice.
Constructed on either a rectangular or a square lattice, pattern class \textit{p2mg} presents two alternating and parallel reflection axes intersecting at right angles with parallel glide-reflection axes. Centres of two-fold rotation are found on the glide-reflection axes positioned at the centre of the unit cell, at each of the unit corners and the midpoints of the unit sides. As shown in Figure 14, the generating region of pattern class \textit{p2mg} is equal to one-quarter of the unit cell area.

Figure 13: Example of a \textit{p2gg} all-over pattern
All-over pattern class *c2mm* is based on a rhombic, a square or a hexagonal lattice structure. Parallel reflection axes and glide-reflection axes alternate with each other in both horizontal and vertical directions, as shown in Figure 15. Centres of two-fold rotation fall on the intersection points of both reflection axes and glide-reflection axes, with the fundamental region consisting of one-quarter of the centred cell. The rhombic-centred cell can generate the pattern by translation alone; however convention states that the larger outline cell should be the generating unit [Hann and Thomson, 1992, p. 40].
4.4 Patterns exhibiting three-fold rotation

There are three all-over pattern classes in which the highest order of rotation is 3 (rotation through 120 degrees). These pattern classes are all constructed on an hexagonal lattice with unit cells bounded by two equilateral triangles. The characteristics of classes \( p_3 \), \( p_{3m1} \) and \( p_{31m} \) are outlined briefly below.

All-over pattern class \( p_3 \), has a unit cell which contains six distinct centres of three-fold rotation, which are located at the unit corners and the centres of the triangular cell components. As shown in Figure 16, the area of the fundamental region is one-third of the unit cell area.

All-over pattern class \( p_{3m1} \) is based on a hexagonal lattice, and is produced through a combination of three-fold rotational centres and reflection axes. As indicated in Figure 17, the fundamental region is
one-sixth of the unit cell. Centres of three-fold rotation are positioned at the intersections of the reflection axes, which are present along the longest diagonal of the unit cell and alternate with glide-reflection axes in all three directions.

Figure 16: Example of a $p3$ all-over pattern

Pattern class $p31m$, illustrated in Figure 18, is also constructed on an hexagonal lattice. Reflection axes are located on each side of the unit cell and also along the shortest diagonal of the cell. Centres of three-fold rotation occur at the centres of the two triangular units and also at the corners of the rhomboid cell where the reflection axes intersect. The fundamental region is equal to one-sixth of the area of the unit cell.
Figure 17: Example of a $p3m1$ all-over pattern

Figure 18: Example of a $p31m$ all-over pattern
4.5 Patterns exhibiting four-fold rotation

There are three classes of all-over patterns which exhibit a highest order of rotation of 4 (90 degrees rotational symmetry). These are classes \( p_4 \), \( p_{4mm} \), and \( p_{4gm} \), all of which are based on a square lattice structure.

All-over pattern class \( p_4 \), constructed using a square unit cell, exhibits no reflection or glide-reflection, presenting only two- and four-fold rotation. Centres of four-fold rotation are located at the centre and corners of the unit cell, with two-fold rotational centres present at the midpoint of each side. As shown in Figure 19, the fundamental region is one-quarter of the unit cell area.

![Figure 19: Example of a \( p_4 \) all-over pattern](image)

The square lattice also provides the construction base for pattern class \( p_{4mm} \), which is generated through reflection of the fundamental region, equal to one-eighth of the unit cell area. Reflection axes are
present at the sides of the unit at mid-way points horizontally and vertically and also run diagonally across the unit cell dividing it into eight equal parts. Centres of four-fold rotation are located at the corners and centre of the unit cell. Two-fold rotational centres are positioned at the midpoint of each unit edge and are intersected at right angles by axes of glide-reflection. The structure of a $p4mm$ pattern class is illustrated in Figure 20.

![Figure 20: Example of a p4mm all-over pattern](image)

Pattern class $p4gm$ is based on the square lattice structure and is generated through reflection and four-fold rotation of a fundamental region occupying one-eighth of the unit cell area. Centres of four-fold rotation are located at the centre and also at the corners of the unit cell. Reflection axes intersect the unit cell at right angles on two-fold rotation centres at the midpoint of each unit side. Glide-reflection axes intersect the reflection axes at 45 degrees and 90 degrees, as shown in Figure 21.
4.6 Patterns exhibiting six-fold rotation

There are two remaining all-over pattern classes in which the highest order of rotational symmetry is 6 (60 degrees rotation). The patterns are constructed using an hexagonal lattice unit bounded by two equilateral triangles, as previously seen with patterns exhibiting three-fold rotation.

All-over pattern class p6, illustrated in Figure 22, exhibits six-fold rotation points at each corner of the hexagonal lattice unit cell. Centres of three-fold rotation are located at the centres of the triangular cells, and centres of two-fold rotation are present at the midpoints of the triangular cell edges. All the six-fold rotational points have the same orientation; the three-fold rotational points have two different orientations; conversely, two-fold rotational centres have three different orientations. The generating region for this pattern class is one-sixth of the unit cell area.
Pattern class $p6mm$ is based on an hexagonal lattice and exhibits two-, three- and six-fold rotation combined with reflection and glide-reflection. As shown in Figure 23, centres of six-fold rotation are located at each corner of the unit cell. Reflection axes connect each corner with the other three and also bisect the opposite unit side at a right angle. Centres of two- and three-fold rotational symmetry are located at the intersection points of reflection axes. The fundamental region is one-twelfth of the unit cell area and is bounded by reflection axes connecting centres of two-, three- and six-fold rotation.
Figure 23: Example of a $p6\text{mm}$ all-over pattern
5. Polyhedra: Symmetry in Three-dimensions

A polyhedron has been defined by Coxeter as:

…a finite, connected set of plane polygons, such that every side of each polygon belongs also to just one other polygon, with the provision that the polygons surrounding each vertex form a single circuit.

[Coxeter, 1948, p.4]

The polygons that join to form polyhedra are called faces, these faces meet at edges, and edges come together at vertices. The polyhedron forms a single closed surface, dissecting space into two regions, the interior, which is finite, and the exterior that is infinite [Coxeter, 1948, p.5].

The regularity of polyhedra involves regular faces, equally surrounded vertices and equal solid angles [Coxeter, 1948, p.16]. Under these conditions, there are nine regular polyhedra, five being the convex Platonic solids and four being the concave Kepler-Poinsot solids [Cromwell, 1997, p.53]. The focus of attention in this monograph is with the former group, the Platonic solids. These are as follows: the (regular) tetrahedron, the (regular) octahedron, the cube (or regular hexahedron), the (regular) icosahedron and the (regular) dodecahedron, as illustrated in Figure 24.
A convex polyhedron is called *semi-regular* if its faces have a similar arrangement of regular polygons and equal vertices but is formed from two or more types of polygon [Holden, 1991, p.41]. There are a total of thirteen semi-regular polyhedra, more commonly known as the Archimedean solids, as shown in Figure 25. Archimedes (260-212 BCE) was the first to enumerate these solids although his original work is thought to have been lost in the fire at the Royal Library of Alexandria, Egypt [Wenninger, 1971, p.2].
The regular polyhedra have a certain mathematical elegance. Their highly symmetrical forms give them an aesthetic quality that interests both mathematicians and those less mathematically inclined. Rich in connections to the disciplines of art, architecture and science, the Platonic solids have practical applications in architecture, and occur in nature as crystals and viruses. Artists and sculptors have also created imaginative extensions. The tetrahedron, cube and octahedron occur in nature as sodium sulphantimoniate, sodium chloride and chrome alum respectively, but the more complex solids of the dodecahedron and icosahedron cannot it seems occur as crystals (although crystals of pyrite resemble distorted dodecahedra) [Steinhaus, 1999, pp.207-208]. Ernst Haeckel [1904] observed the icosahedra and dodecahedra in the skeletons of microscopic sea animals radiolaria, as shown in Figure 26. The most perfect examples of these are the Circogonia icosahedra and Circorrhegma dodecahedra radiolaria [Coxeter, 1948, p.13; Thompson, 1961, p.726]. In Plato’s Timaeus, four of the solids were related to the four elements: earth, fire, water and air, with the dodecahedron representing the cosmos [Plato, 1961]. These mathematical solids served as Kepler’s models for the orbits of planets and also formed the basis of Buckminster Fuller’s geodesic domes [Kappraff, 1986].

![Figure 26: Icosahedral- and dodecahedral-shaped radiolaria](source: Reproduced from Kappraff [1986, p.919])
The ancient Greeks are usually credited with being the first to discover the regular polyhedra, and are believed to have composed the earliest written records and mathematical descriptions of the solids. There is however evidence that they were known long before the time of the Greeks. An Etruscan dodecahedron made of soapstone, discovered near Padua (in Northern Italy) in the late 1800s, dates back more than 2,500 years [Martin, 1982, p.199]. Around fifty hollow bronze dodecahedra dating from Roman times have been discovered throughout the northwestern provinces of the Roman empire [Malkevitch, 1988; Artmann, 1993]. These solids are embellished with spheroids at each vertex and with circular holes on each face. Artmann [1996] also reports on a bronze Roman icosahedron. Predating the Romans and the Etruscans come discoveries that the Neolithic people of Scotland constructed hundreds of spherical stone models, some of which exhibit carved edges that correspond to the five Platonic solids, dating from perhaps 4000 years ago [Atiyah and Sutcliffe, 2003]. Examples of these stones are kept in the John Evans room at the Ashmolean Museum in Oxford. Further details can be found in Smith [1874-76] and Marshall [1976-77].

The root of the human discovery of the Platonic solids is impossible to trace. There is no proof as to whether or not the ancient Scots or the Etruscans had any mathematical understanding of the regular solids. It is from the ancient Greeks that the knowledge of the regular and semi-regular polyhedra has been passed down and it is this source that has inspired the modern mathematical treatment of them.

The regular solids were constructed by Plato (427-347 BCE) in around 400 BCE [Cromwell, 1997,p.51]. Before him the Pythagoreans associated the tetrahedron, cube, octahedron and icosahedron with the four elements, and regarded the dodecahedron as a shape enveloping the universe [Coxeter, 1948, p.13; Heath, 1981, p.158]. The abstract concept of a regular solid is due to Theætetus of Athens (419-369 BCE), who provided mathematical descriptions for all five [Martin, 1982, p.199; Heath, 1981, p.294-5]. The Platonic solids featured predominately in the philosophy of Plato, a contemporary of Theætetus, who wrote the oldest surviving discussion of them in the dialogue Timæus c.360 BCE.
[Cromwell, 1997, p.51]. It is from Plato’s name that the term Platonic solid is derived. Theætetus is also famous for laying the foundation of the study of irrationals that appears in Books XIII - XV of Euclid’s Elements (Books XIV - XV were not written by Euclid himself but by several authors at a later date) [Heath, 1981, p.419; Holden, 1991, p.1]. Propositions thirteen to seventeen of Book XIII describe the construction of the tetrahedron, octahedron, cube, icosahedron and dodecahedron respectively. In proposition eighteen, Euclid argued the proof that there are only five regular convex solids [Heath, 1981, p.419].

Pappus of Alexandria, in Book V of his Mathematical Collection [Pappus, 1939, p.195], attributed the discovery of the thirteen semi-regular polyhedra to Archimedes. Although Archimedes’ original manuscript on the semi-regular solids was lost, these thirteen solids were named the Archimedean polyhedra in tribute. German astronomer, Johannes Kepler (1571-1630) rediscovered the semi-regular solids, publishing a complete list of all thirteen and assigning them the names by which they are still known [Cromwell, 1997, p.81]. Seven of the Archimedean solids can be obtained from the Platonic solids by slicing off either vertices or edges with a cutting plane. This process is known as truncation. The truncated octahedron, truncated cube and cuboctahedron can be obtained by truncating either the cube or the octahedron. The truncated icosahedron, truncated dodecahedron and the icosidodecahedron are related to the icosahedron and the dodecahedron. Only one Archimedean solid can be obtained from the tetrahedron, this is the truncated tetrahedron. The rhombicuboctahedron, rhombicosidodecahedron, great rhombicuboctahedron and the great rhombicosidodecahedron are produced through expansion of a Platonic or an Archimedean solid. The expansion of a polyhedron radially displaces the faces of the solid, whilst keeping their size and orientation constant, and filling the gaps produced with new faces [Ball and Coxeter, 1987, pp.139-140]. The remaining two snub polyhedra, the snub cube and the snub dodecahedron, are related to the cube and the dodecahedron and are produced through a process of snubification, which involves the expansion and twisting of faces.
The regular and semi-regular polyhedra were rediscovered during the Renaissance, although written description of the Archimedean solids did not emerge until the early-seventeenth century [Malkevitch, 1988; Kapraff, 1991, p.328]. Italian Renaissance artists such as Paolo Uccello (ca. 1397-1475) and Piero della Francesca (ca. 1420-1492) were fascinated with geometry and used polyhedra to demonstrate their mastery of linear perspective. A particularly influential Renaissance publication was *De Divina Proportione* [1509] by Luca Pacioli (ca. 1445-1517), which incorporated illustrations by Leonardo da Vinci (1452-1519). These illustrations were probably the first to show polyhedra as three-dimensional lattice-type structures [Tomlow, 1997].

The German Renaissance print-maker, Albrecht Dürer’s (1471-1528) interest in polyhedra is evident from his copperplate engraving *Melancholia* (1514), in which an angel contemplates a truncated stone polyhedron. Dürer made a significant contribution to Renaissance literature with his highly regarded publication *Unterweysung der Messung mit dem Zirkel und Richtscheit* (Treatise on Measurement with the Compass and Ruler) (1525), which was one of the early publications to teach methods of perspective. In this book Dürer also presented the earliest known examples of polyhedral nets, which are plane diagrams showing the edges of polyhedra [Weisstein, 2004].

In the fifteenth century, knowledge of the next generation of regular polyhedra began to emerge. Kepler endeavoured to unify the disciplines of geometry, music, astrology and astronomy, developing a polyhedral model of the structure of the universe [Cromwell, 1997, p.141]. From this research came the discovery of the Kepler-Poinsot solids and Kepler’s laws of planetary motion, for which he is now famous. Kepler’s figures from *Harmonice Mundi* [1619], better known for containing his third law of planetary motion, also illustrated two new polyhedra, the small stellated dodecahedron and great stellated dodecahedron. These regular star polyhedra are composed of concave regular polygons, either as faces or vertex figures (circuits joining the midpoints of faces around each vertices) and are constructed from the Platonic solids through a process called stellation. This process extends the facial planes of a polyhedron, past the polyhedron edges, until the
planes intersect [Wenninger, 1989]. Kepler was the first to recognise that these polyhedra could be considered ‘regular’ if he removed the restriction that regular polytopes must be convex. Kepler’s stellated solids were rediscovered by Louis Poinsot (1777-1859), in 1819, along with two other non-convex regular solids, the great icosahedron and great dodecahedron [Poinsot, 1819; Wells, 1991, p.130].

It was not until the nineteenth century that the regular polytopes in higher dimensions were examined and characterised by Swiss mathematician, Ludwig Schläfli [1901]. A comprehensive account of the work of Schläfli and others in this area is provided by Coxeter [1948]. Coxeter and others also further advanced the study of the stellations of the Platonic solids in 1938, with the publication of the famous paper The Fifty-nine Icosahedra [Coxeter et al, 1999].

5.1 The symmetry characteristics of the Platonic solids
A polyhedron can be defined as a three-dimensional solid that consists of a collection of polygons, usually joined at their edges. As noted previously, the polygons that join to form polyhedra are called faces. These faces meet at edges, and the edges meet at vertices. Composed of faces identical in size and shape, equally surrounded vertices and equal solid angles, the regular convex solids, known as the Platonic solids, are highly symmetrical polyhedra. The symmetry characteristics of reflection and rotation that govern the properties of regularly repeating patterns and tilings, are also of importance to three-dimensional solids. It has been noted there are exactly five regular convex polyhedra: the tetrahedron, octahedron, cube, dodecahedron and icosahedron, as proved by Euclid in the last proposition of the Elements [Heath, 1981, p.419].

Only three types of regular polygons are found in the regular solids. The cube is composed of six squares and the dodecahedron of twelve pentagons. The tetrahedron, octahedron and icosahedron are comprised of four, eight and twenty equilateral triangles respectively. It is not possible to create a polyhedron composed of hexagonal faces or from polygons with any greater number of sides [Wenninger, 1971, p.2]. The uniformity of the faces of any regular polyhedron may be
relinquished causing only distortion of the polyhedron [Coxeter, 1948, p.5].

The Platonic solids also exhibit duality, a characteristic that pairs them with one another. The cube and the octahedron are considered to be duals; by marking and connecting a point in the centre of each of the cube’s six faces the outline of an octahedron is drawn. By doing the same to an octahedron a cube is outlined, as shown in Figure 27. Points in the centre of the faces of a dodecahedron can be connected to outline an icosahedron, with the twelve faces and twenty corners of a dodecahedron becoming the twelve corners and twenty faces of an icosahedron. The dual relationship between the dodecahedron and the icosahedron is shown in Figure 28. Interchanging the corners and faces of the tetrahedron produces another tetrahedron, as shown in Figure 29. The tetrahedron is therefore a self-dual as the two are geometrically similar. The duality of the solids also demonstrates where the axes of reflection lie in relation to each other [Holden, 1991, p.14].

Figure 27: The inscribed cube and octahedron duals
The ancient Greeks were aware of the duality of the regular solids as in around 300 BCE, the unknown author of Euclid’s *Elements* XV, inscribed an octahedron in a cube, a cube in an octahedron and a dodecahedron in an icosahedron [Coxeter, 1948, p.30].

The Platonic solids are related to each other in many ways aside from their duality. Euclid, in Book XIII of *Elements*, showed how the solids were associated through the *golden mean*. In any convex solid, a theorem of Euclid also explains that the angles at any vertices must amount to less than 360 degrees, with the angle calculation often falling considerably short of 360 degrees. René Descartes (1596-1650) proved that the angular defect when summed for all vertices always totals 720 degrees [Wenninger, 1971, p.1]. The polyhedral formula, discovered independently by Euler (in 1752) and Descartes, relating to the number of vertices, faces and edges of a polyhedron states: $V + F - E = 2$. Although this formula does not stand for the stellated polyhedra [Steinhaus, 1999, pp.252-253].
The Schläfli notation \{p,q\} can be used to describe a polyhedron, whose faces are \(p\)-gons (denoted by \{p\}), with \(q\) the number of faces meeting at each polyhedron vertex. The dihedral angle is the internal angle formed between the planes of two adjoining polygons [Pearce, 1978, p.35]. The dihedral angles of the Platonic solids are all equal; however, of the Archimedean solids only the cuboctahedron and the icosidodecahedron have equal dihedral angles [Williams, 1979, p.74 and p.86]. The dihedral angles of polyhedra become an important factor when considering space-packing polyhedra. The more important symmetry characteristics of the five Platonic solids are listed in Table 2.

Table 2: The symmetry characteristics of the Platonic solids

<table>
<thead>
<tr>
<th>Platonic solid</th>
<th>faces</th>
<th>edges</th>
<th>vertices</th>
<th>Axes of rotation</th>
<th>Faces of reflection</th>
<th>Schláfli symbol</th>
<th>Dihedral angle</th>
</tr>
</thead>
<tbody>
<tr>
<td>tetrahedron</td>
<td>4</td>
<td>6</td>
<td>4</td>
<td>3</td>
<td>4</td>
<td>6</td>
<td>{3,3}</td>
</tr>
<tr>
<td>octahedron</td>
<td>8</td>
<td>12</td>
<td>6</td>
<td>6</td>
<td>4</td>
<td>3</td>
<td>{3,4}</td>
</tr>
<tr>
<td>cube</td>
<td>6</td>
<td>12</td>
<td>8</td>
<td>6</td>
<td>4</td>
<td>3</td>
<td>{4,3}</td>
</tr>
<tr>
<td>dodecahedron</td>
<td>12</td>
<td>30</td>
<td>20</td>
<td>15</td>
<td>10</td>
<td>6</td>
<td>{5,3}</td>
</tr>
<tr>
<td>icosahedron</td>
<td>20</td>
<td>30</td>
<td>12</td>
<td>15</td>
<td>10</td>
<td>6</td>
<td>{3,5}</td>
</tr>
</tbody>
</table>


Polyhedra may be drawn as plane diagrams, known as nets, in which the faces and edges of the polyhedron are shown. These may be considered as unfolded polyhedra. The net of a polyhedron may specify which edges are to be joined. There may be several possible nets that represent a single polyhedron. The cube, for example, has eleven possible nets, as shown in Figure 30. The number of possible
nets for the Platonic solids is listed in Table 3. It should be noted that the Platonic duals have the same number of possible unfoldings.

![The eleven possible nets for the cube](image)

**Figure 30: The eleven possible nets for the cube**

<table>
<thead>
<tr>
<th>Platonic solid</th>
<th>Number of nets</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tetrahedron</td>
<td>2</td>
</tr>
<tr>
<td>Octahedron</td>
<td>11</td>
</tr>
<tr>
<td>Cube</td>
<td>11</td>
</tr>
<tr>
<td>Dodecahedron</td>
<td>43380</td>
</tr>
<tr>
<td>Icosahedron</td>
<td>43380</td>
</tr>
</tbody>
</table>

*Source: data from Weissten [2004b]*

When considering the symmetries of the Platonic solids there are three possible groups of polyhedral symmetry. The *icosahedral* symmetry group, as illustrated by the icosahedron and its dual the dodecahedron; the *octahedral* symmetry group, illustrated by the octahedron and its dual the cube; and the *tetrahedral* symmetry group, illustrated by the self-dual tetrahedron. Gerretsen and Verdenduin [1974] determined the symmetry groups of all polyhedra with respect to their vertices; those of the Platonic solids are listed in Table 2 in association with their Schlüfli notation. For further information on the polyhedral symmetry groups, see Cromwell [1997, chp.8].
Descriptions of the symmetry properties of each individual solid are provided in the sections below, together with relevant illustrative material. A key for the diagrams illustrating the symmetry characteristics of the Platonic solids is given below.

Key:
- plane of reflection
- axis of two-fold rotation
- axis of three-fold rotation
- axis of four-fold rotation
- axis of five-fold rotation

### 5.2 The tetrahedron

The tetrahedron is a pyramid on a triangular base and the simplest of the Platonic solids [Coxeter, 1948, p.4]. The tetrahedron consists of four equilateral triangular faces, four vertices and six edges. Each of the four vertices is surrounded by three triangles, which produces a vertex angle (sum of the vertex angles of the polygons meeting at each polyhedron vertex) of 180 degrees [Wenninger, 1971, p.1]. The tetrahedron has seven axes of rotational symmetry: four axes of three-fold rotation connect each vertex with the midpoints of the opposite faces, and three axes of two-fold rotation pass through midpoints of the opposite edges. In addition to rotational symmetry, the tetrahedron possesses six planes of reflection passing through axes of two-fold rotation and the edges of the tetrahedron. As noted previously, the tetrahedron is its own dual polyhedron and therefore connecting the centres of the faces of a tetrahedron forms another tetrahedron. The symmetry characteristics of the tetrahedron are illustrated in Figure 31 and the relevant net for the tetrahedron is shown in Figure 32.

![Figure 31: Reflection and rotation in the tetrahedron](image-url)
5.3 The cube

The regular hexahedron, more commonly known as the cube, consists of six square faces that meet at right angles, any of which may be regarded as the base. A total of eight vertices and twelve edges are evident, with each vertex surrounded by three square polygon faces, resulting in a vertex angle of 270 degrees [Wenninger, 1971, p.1]. The cube possesses thirteen axes of rotational symmetry. Six axes of two-fold rotation pass through the centres of opposite edges and four axes of three-fold rotation coincide with the diagonals of the cube, connecting its opposite vertices. Axes of four-fold rotation also connect the midpoints of opposite faces, with all axes of rotation passing through the centre of symmetry. The existence of a centre of symmetry in the cube is shown by the fact that each face has a corresponding equal and parallel face [Shubnikov and Koptsik, 1974, p.56]. The cube also exhibits a total of nine planes of reflection, illustrated in Figure 33. A net for the cube is shown in Figure 34.

Figure 32: A net for the tetrahedron

Figure 33: Reflection and rotation in the cube
5.4 The octahedron

The octahedron may be regarded as a dipyramid, formed by placing two equal square-based pyramids base to base [Coxeter, 1948, p.5], as seen in Figure 35. The octahedron is composed of eight equilateral triangular faces, twelve edges and six vertices. Four triangular faces surround each vertex, which results in a vertex angle of 240 degrees [Wenninger, 1971, p.1]. The octahedron displays six axes of two-fold rotation passing through the midpoint of opposite edges, four axes of three-fold rotation connecting the centre of opposite faces and three axes of four-fold rotation joining opposite vertices. In addition to rotational symmetry, the octahedron exhibits nine planes of reflection. The octahedron possesses the same symmetry characteristics as the cube. As the dual of the cube, the centres of the faces of the octahedron can be connected to form a cube, and vice versa [Steinhaus, 1999, p.194] The dual relationship of the octahedron and the cube is shown in Figure 27. An illustration of a net for the octahedron is shown in Figure 36.
5.5 The dodecahedron

The dodecahedron is composed of twelve pentagonal faces, thirty edges and twenty vertices. Each vertex is surrounded by three pentagonal faces, displaying vertex angles of 216 degrees [Wenninger, 1971, p.1]. The dodecahedron exhibits thirty-one axes of rotational symmetry. Fifteen axes of two-fold rotation pass through the midpoint of opposite edges, ten axes of three-fold rotation connect opposite vertices, and six axes of five-fold rotation link the centres of opposite faces. In addition to rotational symmetry the dodecahedron also exhibits fifteen planes of reflection, as illustrated in Figure 37. A dodecahedral net is shown in Figure 38.
5.6 The icosahedron
The icosahedron consists of twenty equilateral triangular faces. A total of twelve vertices and thirty edges are present on the solid. Each vertex is surrounded by five equilateral triangles producing a vertex angle of 300 degrees [Wenninger, 1971, p.1]. As dual of the dodecahedron, the icosahedron also displays the same set of symmetries as its dual the dodecahedron. Fifteen axes of two-fold rotation join the midpoints of opposite edges. Ten axes of three-fold rotation pass through the centres of opposite faces, and six axes of five-fold rotation connect opposite vertices. Comparable with the dodecahedron, the icosahedron also exhibits fifteen planes of reflection. These symmetry characteristics are illustrated in Figure 39 and an icosahedral net is shown in Figure 40.
Figure 39: Reflection and rotation in the icosahedron

Figure 40: A net for the icosahedron
6. **Patterning Polyhedra: The Application of All-over Patterns to the Platonic Solids**

The patterning of regular solids in ways that ensure precise registration and the absence of gaps or overlaps is neither a trivial nor a straightforward matter. When all-over patterns, which perform to satisfaction on the Euclidean plane, are folded several times, into different planes, their component parts will not correspond readily. Only certain pattern types, with particular symmetry characteristics, are suited to the precise patterning of each Platonic solid. This section presents a systematic means by which appropriate pattern types can be identified. The difficulties encountered in the patterning process are explained, and the symmetry characteristics of importance to the process are identified.

6.1 **Lattice structures**

It has been established in earlier sections that all-over patterns exhibit symmetries, which combine to produce seventeen possibilities across the plane. Proof of the existence of only seventeen all-over pattern classes is provided by Weyl [1952], Coxeter [1969] and Martin [1982]. Designs possessing the same symmetry combinations are said to belong to the same symmetry class, and may be classified accordingly. Further accounts of the classification and construction of all-over patterns were given by Woods [1935], Schattschneider [1978], Stevens [1984], Washburn and Crowe [1988] and Hann and Thomson [1992].

A further geometrical element of importance to pattern structure is the underlying framework or lattice, as noted previously. Each lattice (of which there are five distinct types) is comprised of unit
cells of identical size, shape and content, which contain the essential repeating unit or element of the pattern or tiling, as well as the symmetry instructions for the pattern's construction. In this section identification is made of particular all-over-pattern classes which are capable of patterning the Platonic solids without gap or overlap. In order to assess the geometric characteristics of importance in this process it seems to be best to focus on the application of areas of the unit cell (capable of building-up the repeating pattern) and to allow these areas to act as tiles when applied to the faces of the polyhedra. The emphasis is thus placed on the pattern's underlying lattice structure and the symmetry operations contained within it.

The first step in patterning a regular polyhedron involves matching the polyhedral faces to a suitable lattice type. The polygons that comprise the faces of the Platonic solids, are the equilateral triangle, the square and the pentagon. The seventeen all-over pattern classes may be constructed from either a square or an hexagonal lattice and some may be constructed from both lattice types as indicated previously in Table 1 [reproduced from Tantiwong, 2000, p.27]. Suitable patterns applicable to the tetrahedron, octahedron and icosahedron, whose faces are equilateral triangles, must therefore be constructed on a hexagonal lattice, where the unit cell comprises two equilateral triangles. The cube is the only Platonic solid that is composed of square faces. All pattern classes constructed on a square lattice are therefore suited to patterning the surface of the cube. Patterning the dodecahedron, composed of regular pentagonal faces, requires a different approach. It is well known that the regular pentagon, with five-fold rotational symmetry, cannot tile the plane without gap or overlap. There are, however, fourteen (known) types of equilateral pentagons that can tessellate the plane [Wells, 1991, pp.177-179]. Probably the best known is the Cairo tessellation (Figure 41), formed by convex equilateral pentagons (equal-length sides, but different associated angles). Using knowledge of the Cairo tessellation the method used by Schattschneider and Walker [1982, p.26] presents a means by which to tile the dodecahedron. This is explained further in the appropriate sub-section below.
All pattern classes constructible on an hexagonal lattice were systematically applied to the tetrahedron, octahedron and icosahedron. Initial constructions utilised half the hexagonal unit cell (equivalent to an equilateral triangle) to pattern each face. All pattern classes based on a square lattice were systematically applied to the cube in a similar manner, utilising an area equal to the whole unit cell. Further constructions involved extracting smaller areas of the unit cell for use as tiles. The area constitutes either a square or equilateral triangle and must be capable of creating the full repeating pattern when symmetries are applied. The smaller the area of the pattern used to tile the polyhedron, the larger the scale of the pattern on the solid. Further description, discussion and illustration is provided below.

Table 4 identifies the all-over classes capable of patterning the regular solids along with their constituent lattice structures and the area of unit cell applied to each face. The symmetry characteristics of these regularly patterned solids are described further below.
Table 4: The symmetry characteristics exhibited by the regularly patterned Platonic solids

<table>
<thead>
<tr>
<th>Platonic solid</th>
<th>Pattern class</th>
<th>Lattice structure</th>
<th>Area of unit cell on face</th>
<th>Equivalent vertices</th>
<th>Rotation present on vertices</th>
<th>Rotation present on edges</th>
<th>Rotation present on faces</th>
<th>Reflection present on edges</th>
<th>Reflection present on faces</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tetrahedron</td>
<td>p2</td>
<td>hex.</td>
<td>½</td>
<td>✓</td>
<td>-</td>
<td>2-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>c2mm</td>
<td>hex.</td>
<td>½</td>
<td>✓</td>
<td>-</td>
<td>2-</td>
<td>-</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td></td>
<td>p6</td>
<td>hex.</td>
<td>½ ✓</td>
<td>3-</td>
<td>2-</td>
<td>3-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>p6mm</td>
<td>hex.</td>
<td>½ ✓</td>
<td>3-</td>
<td>2-</td>
<td>3-</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Octahedron</td>
<td>p3</td>
<td>hex.</td>
<td>½ ✓</td>
<td>2-</td>
<td>-</td>
<td>3-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>p31m</td>
<td>hex.</td>
<td>½ ✓</td>
<td>2-</td>
<td>-</td>
<td>3-</td>
<td>✓</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>p3m1</td>
<td>hex.</td>
<td>½ ✓</td>
<td>2-</td>
<td>-</td>
<td>3-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>p3m1</td>
<td>hex.</td>
<td>½ -</td>
<td>a) 2-</td>
<td>b) 2-</td>
<td>c) 2-</td>
<td>-</td>
<td>✓</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>p6</td>
<td>hex.</td>
<td>½ ✓</td>
<td>4-</td>
<td>2-</td>
<td>3-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>p6mm</td>
<td>hex.</td>
<td>½ ✓</td>
<td>4-</td>
<td>2-</td>
<td>3-</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td></td>
<td>p6mm</td>
<td>hex.</td>
<td>½ -</td>
<td>a) 4-</td>
<td>b) 2-</td>
<td>-</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td></td>
<td>p6</td>
<td>hex.</td>
<td>½ ✓</td>
<td>5-</td>
<td>2-</td>
<td>3-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>p6mm</td>
<td>hex.</td>
<td>½ ✓</td>
<td>5-</td>
<td>2-</td>
<td>3-</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Icosahedron</td>
<td>p4</td>
<td>sq.</td>
<td>1</td>
<td>✓</td>
<td>3-</td>
<td>2-</td>
<td>4-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>p4mm</td>
<td>sq.</td>
<td>1 ✓</td>
<td>3-</td>
<td>2-</td>
<td>4-</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td></td>
<td>p4gm</td>
<td>sq.</td>
<td>1 ✓</td>
<td>3-</td>
<td>2-</td>
<td>4-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Cube</td>
<td>p4</td>
<td>sq.</td>
<td>½ ✓</td>
<td>3-</td>
<td>2-</td>
<td>5-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Dodecahedron</td>
<td>p4</td>
<td>sq.</td>
<td>½ ✓</td>
<td>3-</td>
<td>2-</td>
<td>5-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>
6.2 Patterning the tetrahedron

Patterning of the tetrahedron is possible with certain patterns possessing two-fold and six-fold rotational symmetry. An area equivalent to half the unit cell forms the repeating unit in each case. When a \( p2 \) or \( c2mm \) pattern is applied to the tetrahedron (as shown in Figures 42 – 43 and 44 – 45 respectively), the pattern exhibits no rotation at the vertices but tetrahedral axes of two-fold rotation are maintained through the mid-point of opposite edges. The edges of the solid patterned with class \( c2mm \), however, are not equivalent as two edges exhibit planes of reflection, which slice through the relevant edge and divide the solid into two equal parts. A reflection plane therefore runs through the centre of each face.

![Figure 42: Patterning of the tetrahedron using pattern class \( p2 \)](image)

![Figure 43: The tetrahedron patterned with class \( p2 \)](image)
Tetrahedra patterned with patterns possessing six-fold rotation maintain the original rotational properties of the solid. Centres of six-fold rotation, become axes of three-fold rotation at each vertex on the regularly patterned solid. Centres of two-fold and three-fold rotation in the pattern remain unchanged when applied to the tetrahedron; they locate at the midpoint of each edge and the centre of each face respectively. A tetrahedron tiled with class p6mm also presents the six reflection planes of the tetrahedron. Figures 46 – 47 and 48 – 49 show, respectively, illustrations of pattern classes p6 and p6mm patterning the tetrahedron, in which areas equal to half the unit cell tile each face.
Figure 46: Pattern of the tetrahedron using pattern class $p_6$

Figure 47: The tetrahedron patterned with pattern class $p_6$
In the case of the tetrahedron, it seems that a pattern must possess reflection and rotational properties equal to, or higher than those of the tetrahedron, in order to maintain the rotational symmetries of the solid. Pattern classes \( p_3 \), \( p_3m1 \) and \( p_31m \) are not applicable to the tetrahedron due to the two types of tile created from the equilateral triangular halves of the unit cells and the reflection axes created in class \( p_31m \). For these patterns to repeat around the surface of a polyhedron there must be an even number of faces meeting at each vertex.
6.3 Patterning the cube

Connecting points of four-fold rotation in plane pattern classes \( p_4 \), \( p_{4gm} \) and \( p_{4mm} \) produces a square grid, which will lend itself readily to the patterning of the cube. The application of these pattern classes to the cube maintains cubic rotational symmetry; three-fold rotation is preserved at the vertices, two-fold rotation at the mid-point of each edge and four-fold rotation at the centre of each face. A cube patterned using class \( p_{4mm} \) also retains the reflection symmetry of the cube with a total of nine reflection planes slicing through each face and each of the twelve edges. Illustrations of the cube regularly patterned with pattern classes \( p_4 \), \( p_{4gm} \) and \( p_{4mm} \) are shown in Figures 50 – 51, 52 – 53, and 54 – 55, respectively.

Figure 50: Patterning of the cube using pattern class \( p_4 \)

Figure 51: The cube patterned with pattern class \( p_4 \)
Figure 52: Patterning of the cube using pattern class $p4gm$

Figure 53: The cube patterned with pattern class $p4gm$
6.4 Patterning the octahedron

Pattern classes p3, p31m, p3m1, p6 and p6mm are each suited to patterning the octahedron. Relevant illustrations are provided in Figures 56 – 57, 58 – 59, 60 – 61, 62 – 63 and 64 – 65 respectively. These possibilities incorporate repeating components of areas equivalent to half and one-sixth that of the unit cell. When considering components which comprise half the area of the unit cell, and exhibit the highest order of rotation of three, the unit cell must consist of two independent halves or possess reflection planes which correspond with the edges
of the solid, in order to permit the four octahedral faces to meet at each vertex. The two tiles, created by each half of class $p3$ and $p3m1$ unit cells, have two-fold rotation at each vertex but no rotation at the edges, as illustrated in Figures 57 and 59. The octahedron tiled with pattern class $p3m1$ also exhibits six planes of reflection. Pattern class $p31m$ presents axes of two-fold rotation at the vertices as a result of the two intersecting reflection planes which slice through opposite edges. Axes of three-fold rotation, preserving the three-fold rotational symmetry of the solid, are evident at the centre of each face, as shown in Figure 61.

Figure 56: Patterning of the octahedron using pattern class $p3$

Figure 57: The octahedron patterned with pattern class $p3$
Figure 58: Patterning the octahedron using pattern class $p3m1$

Figure 59: The octahedron patterned with class $p3m1$
Pattern classes $p6$ and $p6mm$ maintain the rotational symmetries of the octahedron due to the high symmetry characteristics of the patterns themselves. Four-fold rotation is present at each vertex; two-fold rotation is present at the mid-point of each edge and three-fold rotation at the centre of each face. Class $p6mm$ also retains all the reflection planes of the octahedron. Illustrations of the regular patterning of the octahedron using pattern classes $p6$ and $p6mm$ are shown in Figures 62 – 63 and 64 – 65, respectively. Comparable
with the tetrahedron, it seems that a pattern must possess rotational symmetry characteristics equal to, or higher than those of the solid, in order to preserve the underlying symmetries of the octahedron.

Figure 62: Patterning of the octahedron using pattern class p6

Figure 63: The octahedron patterned with pattern class p6
As stated previously, it is possible to pattern the octahedron using a repeating unit with an area equal to one-sixth of the unit cell of pattern classes \( p_3m_1 \), \( p_6 \) and \( p_6mm \). The use of a small component of the unit cell imparts different characteristics to the octahedron than seen previously. When pattern class \( p_3m_1 \) is applied to the octahedron three different vertices are evident (in terms of pattern), with identical types found in opposite pairs each exhibiting two-fold rotation, as shown in Figures 66 - 67. Patterning with one-sixth of the area of the unit cell of pattern class \( p_3m_1 \) requires an even number of faces to meet at each
vertex in order for reflection at each edge to form the whole unit cell. This unit is then repeated through rotation or successive reflection across three reflection planes to tile the octahedron.

Figure 66: Patterning the octahedron with pattern class \( p3m1 \), where the area used to pattern each face is equal to one sixth of the unit cell.

Figure 67: The octahedron patterned with pattern class \( p3m1 \), where the area used to pattern each face is equal to one sixth of the unit cell.
The application of components equivalent to one-sixth of the area of class \( p6 \) and \( p6mm \) unit cells results in the presence of two different types of vertices. In each case one axis of two-fold rotation and two axes of four-fold rotation are produced. Axes of two-fold rotation are also found at the mid-point of the four edges that connect the vertices exhibiting two-fold rotation. The reflection present in pattern class \( p6mm \) also imparts reflection planes that slice through the centre of each face and through each edge. An octahedron patterned with a class \( p6 \) or \( p6mm \) plane pattern resembles two identically patterned pyramids placed base to base, as illustrated in Figures 68 – 71.

Figure 68: Patterning the octahedron with pattern class \( p6 \), where the area used to pattern each face is equal to one sixth of the unit cell.

Figure 69: The octahedron patterned with pattern class \( p6 \), where the area used to pattern each face is equal to one sixth of the unit cell.
Figure 70: Patterning the octahedron with pattern class *p6mm*, where the area used to pattern each face is equal to one sixth of the unit cell.

Figure 71: The octahedron patterned with pattern class *p6mm*, where the area used to pattern each face is equal to one sixth of the unit cell.

When a component equal to one-sixth of the area of the unit cell is applied to the octahedron, the pattern must be capable of repetition through rotation or reflection around an even number of faces. A pattern possessing higher rotational properties than the octahedron will rotate around the vertices, as is the case when classes *p6* and *p6mm* are used. Patterns possessing lower orders of rotation
must contain reflection axes that correspond with the edges of the octahedron to allow repetition around the four faces meeting at each vertex, as demonstrated by the use of pattern class \(p3m1\) (see Figures 66 and 67).

### 6.5 Patterning the dodecahedron

The regular patterning of the dodecahedron requires a different approach than the other Platonic solids as five-fold rotational symmetry is not possible in the seventeen all-over pattern classes. There are however fourteen types of irregular pentagons that will tessellate the plane. Of these types is the Cairo tiling (also known as the Cairo tessellation) of equilateral pentagons, which are geometrically the closest to a regular pentagon. This may be classified as pattern type \(p4gm\). However, if a motif exhibiting five-fold rotational symmetry is created within each pentagonal tile, the tessellation’s reflection axes are destroyed and the pattern becomes classifiable as the lower symmetrical pattern class \(p4\). As noted previously, pattern type \(p4\) is capable of tiling the cube. When the \(p4\) pattern based on the Cairo tessellation is applied to the cube, exactly twelve pentagons cover its faces (as shown in Figure 72. Due to the interrelationships between the Platonic solids (see Pugh, 1976, p.13 for further explanation), the cube may be inscribed into the dodecahedron so that each edge lies on a face of the dodecahedron and each vertex also falls at a vertex of the dodecahedron (Figure 73). The pattern may then be projected outwards from the cube onto the faces of the dodecahedron with only minor distortion. This method was applied by Schattschneider and Walker [1982, p.26] in their application of M.C. Escher’s tiling design *Shells and Starfish* to the surface of the dodecahedron. Using this method, a net for the regularly patterned dodecahedron was developed, and this shown in Figure 74. It should be noted that the pattern shown in Figures 72 to 74 does not exhibit true five-fold rotation, but simply fills the pentagonal shape as fully as possible.
Figure 72: Patterning the cube with pattern derived from a pattern based on the Cairo tessellation.

Figure 73: The cube inscribed within the dodecahedron and the resultant patterned dodecahedron.

Figure 74: A net for the tiled dodecahedron, based on a pattern, developed from the Cairo tessellation.
6.6 Patterning the icosahedron

Only pattern classes containing six-fold rotation are applicable to the regular patterning of the icosahedron. The icosahedron possesses the same rotational properties found in pattern class \( p6 \) and therefore the icosahedral rotational symmetries are preserved in the patterned solid. In addition to rotational symmetry, the application of pattern class \( p6mm \) also maintains icosahedral reflection properties. Figures 75 – 76 and Figures 77 – 78 illustrate the use of pattern classes \( p6 \) and \( p6mm \) respectively, patterning the icosahedron, where an area equal to half the unit cell patterns each face.

Figure 75: Patterning the icosahedron with pattern class \( p6 \)

Figure 76: The icosahedron patterned with pattern class \( p6 \)
Figure 77: Patterning the icosahedron with pattern class \( p6mm \)

Figure 78: The icosahedron patterned with pattern class \( p6mm \)
7. Summary and Conclusions
The patterning of the regular solids, using knowledge of the seventeen classes of all-over patterns, in ways that ensure regular repetition and precise registration is not a straightforward matter. When regularly repeating all-over patterns are folded into different planes, their component parts may not correspond readily. As is the case in two-dimensions, the patterns rely on the symmetries present to repeat by rotation (at edges and vertices) and/or through successive reflection present at the edges of the solid. This monograph presents and discusses a method by which appropriate pattern types can be identified and illustrates their application to the regular polyhedra.

Only ten of the seventeen all-over pattern classes are applicable to the regular patterning of the Platonic solids. These pattern classes are based only on hexagonal or square structures as determined by the face polygons of the regular solids and therefore the applicable pattern classes. Plane patterns with lower rotational symmetry (for example classes \( p1, p1m1, p1g1, c1m1, p2mm, p2mg \) and \( p2gg \)) cannot be applied to regularly pattern the Platonic solids. Only the tetrahedron and the octahedron can be regularly patterned with patterns possessing lower rotational symmetry than the polyhedron. The symmetry of the resultant patterned polyhedron is lowered and no rotation is present on the faces. Pattern classes with a higher rotational symmetry than that of the polyhedron will retain the rotational symmetries of the solid, as demonstrated when classes \( p6 \) and \( p6mm \) were applied to the tetrahedron, octahedron and icosahedron. The symmetry operation of reflection has less of an influence on a pattern’s suitability to cover a solid and planes are only operational on a regularly patterned solid if reflection is present in the two-dimensional pattern. An all-over
pattern that exhibits no reflection will disguise the reflection planes of the polyhedron. In the majority of cases, the area of a pattern used to cover each face of a polyhedron is equivalent to half or the whole unit cell. Only the octahedron can be regularly patterned using an area smaller than half the area of the unit cell (equal to one-sixth), creating two types of vertices. In these cases, no rotation is exhibited on the patterned polyhedron faces and two types of rotation axes are present at opposite vertices.

In order to pattern the Platonic solids in ways that ensure that the unit cell repeats across the polyhedron in the same manner as in the relevant all-over pattern, it is necessary to recognize the significance of symmetry in both two- and three-dimensions. The importance of the symmetry properties of the polyhedra, particularly the order of the rotation axes present at the vertices, has become readily apparent. This characteristic is determined by the number of faces meeting at each vertex of the solid. A pattern applied to the faces of a polyhedron retains its symmetry properties except for the rotational symmetry that falls on a polyhedron’s vertices. A pattern’s order of rotation, at a solid’s vertices, is dependant on the rotation axes of the solid. The evidence of rotation axes on the faces and edges of the solid and the presence of reflection depend entirely on the symmetry characteristics of the two-dimensional pattern. The reflection symmetry of the solid is often disguised by the application of an all-over pattern, but rotation axes operational at each vertex remain evident, except when an element used is of an area smaller than half that of the unit cell. A regularly patterned polyhedron, therefore, can exhibit different symmetry characteristics to the underlying polyhedral structure.

Where solids are patterned with components equal to half or the whole unit cell, vertices are found to be equivalent in terms of their symmetries. Where a smaller component of the unit cell has been manipulated on the surface of a polyhedron, several types of vertex are evident. For example, multiple vertex types are seen when areas equal to one-sixth of the unit cells of pattern classes \( p3m1 \), \( p6 \) and \( p6mm \) are used to pattern the octahedron. It should also be noted that the octahedron is the only Platonic solid capable of being regularly
patterned with areas of an all-over pattern smaller than half the unit cell.

Only ten of the seventeen classes of all-over patterns can be applied to regularly repeat across the faces of the Platonic solids. These pattern classes are all based on either an hexagonal or a square lattice structure. In order for an all-over pattern to cover the regular solids it must possess symmetry axes of two-fold rotation or higher. This contradicts an earlier study by Pawley [1962], which stated that plane patterns must have a symmetry axis higher than the second order to fit onto polyhedra, as classes \( p2 \) and \( c2mm \) are suited to patterning the tetrahedron. As far as the authors are aware, these are the only two pattern types that exhibit the highest rotational symmetry of order two and are capable of patterning a Platonic solid. When applied to the tetrahedron these patterns are the only classes that do not exhibit rotation at vertices. A summary of the findings from the research associated with this monograph is presented in the Appendix; this may prove to be of particular interest to the specialist reader.

Following the guidelines of Bragg [1949 pp.72-73] and Lockwood and Macmillan [1978 pp.123-131], the regularly patterned Platonic solids have been classified by their point group symmetries. Based on this tabulated information, a series of geometric rules to aid the systematic application of patterns to regular polyhedra, can be proposed. These are presented below.

(i) **Lattice structure:** The tetrahedron, octahedron and icosahedron can only be patterned regularly with patterns based on a hexagonal lattice. The cube can only be regularly patterned with patterns based on a square lattice. Only patterns based on the hexagonal or square lattice structures are applicable to the patterning of the regular polyhedra.

(ii) **Orientation:** When applied to a solid, the unit cell must repeat around the solid in such a way that it fits without gap or overlap.

(iii) **Rotational symmetry:** The Platonic solids can be patterned regularly by designs exhibiting higher rotational symmetry than
the solid without altering the symmetry of the solid. The axis of highest rotational symmetry must be present at the corners of the unit cell allowing the axes to fall on the solid’s vertices. Therefore the tetrahedron, octahedron and icosahedron can be patterned with pattern classes $p_6$ and $p_{6mm}$ which contain higher rotational symmetry than the solids. The cube can be patterned with pattern classes $p_4$, $p_{4gm}$ and $p_{4mm}$, as axes of four-fold rotation fall on the solid’s vertices which possess three-fold rotational symmetry.

(iv) Lower orders of symmetry: A Platonic solid can be patterned regularly with a pattern class containing lower rotational symmetry if the number of faces exhibited by the solid allows for complete repetition of the unit cells. The patterning of the tetrahedron with patterns possessing two-fold rotation only, can be facilitated by ensuring that rotation centres are positioned at the mid-point of each edge of the polyhedron. The four faces of the tetrahedron allow two complete repeats of a hexagonal unit cell across its faces. Therefore, the tetrahedron can be patterned regularly with pattern classes $p_2$ and $c_{2mm}$. The octahedron may be patterned with pattern classes exhibiting the highest rotation axes of order three, due to the even number of faces surrounding each vertex which allow two repetitions of the unit cell around each of the vertices and a total of four repetitions across the solid. Therefore, the octahedron can be patterned regularly with pattern classes $p_3$, $p_{3m1}$ and $p_{31m}$.

(v) Reflection planes: The presence of reflection planes on a tiled solid is determined by the presence of reflection in the plane pattern.

(vi) Size of repeating component compared to area of unit cell: A solid may be patterned regularly using an area smaller than half the unit cell of an all-over pattern. This component or tile must contain the fundamental region of the pattern and must satisfy the rotational symmetry requirements of rules two, three and four. Therefore the octahedron can be patterned with an area equal to one-sixth of the unit cell of pattern classes $p_{3m1}$, $p_6$ and $p_{6mm}$. As mentioned previously, the dodecahedron cannot be patterned with a conventional regularly repeating all-over pattern due to the five-fold rotational symmetry of
each face. Manipulation of the so-called Cairo tessellation (composed of equilateral pentagons) produces a design (or framework of a pattern) which may be applied to the cube, followed by projection onto the faces of the dodecahedron.
References


## Appendix

The symmetry characteristics of all-over patterns, Platonic solids and the regularly patterned solids

<table>
<thead>
<tr>
<th>Polyhedron</th>
<th>Pattern class (unit cell area)</th>
<th>Symmetry of plane pattern</th>
<th>Symmetry of polyhedron</th>
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